



TITLE:

Abstract approach to the Dirac equation (Nonlinear Evolution Equations and Mathematical Modeling)

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CITATION:

Okazawa, Noboru ...[et al]. Abstract approach to the Dirac equation (Nonlinear Evolution Equations and Mathematical Modeling). 数理解析研究所講究録 2009, 1640: 67-84

ISSUE DATE:

2009-04

URL:

<http://hdl.handle.net/2433/140571>

RIGHT:

Abstract approach to the Dirac equation

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Abstract

A new existence and uniqueness theorem is established for linear evolution equations in a separable Hilbert space. The result is applied to the Dirac equation with time-dependent potential.

1. Introduction and statement of the result

In this paper we consider the Cauchy problem for the Dirac equation in $L^2(\mathbb{R}^3)^4$:

$$i \frac{\partial u}{\partial t} + H_D u + V(x)u + q(x, t)u = f(x, t),$$

with $u(\cdot, 0) = u_0 \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$, where H_D is the free Dirac operator, $H^1(\mathbb{R}^3)$ is the usual Sobolev space and $H_1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3); (1 + |x|^2)^{1/2}u \in L^2(\mathbb{R}^3)\}$. We shall show the existence of a unique strong solution under some conditions on potentials V , q and inhomogeneous term f . To do so we employ an abstract approach.

Let $\{A(t); 0 < t < T\}$ be a family of closed linear operators in a *separable* complex Hilbert space X . Then the Dirac equation is regarded as one of linear evolution equations of the form

$$(E) \quad \frac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \text{on } (0, T).$$

So we first establish the existence of a unique strong solution to the Cauchy problem of (E) with initial condition. Now let S be a selfadjoint operator in X , satisfying

$$(1.1) \quad (u, Su) \geq \|u\|^2 \quad \text{for } u \in D(S).$$

Then the square root $S^{1/2}$ is well-defined and $Y := D(S^{1/2})$ is also a separable Hilbert space, with inner product $(u, v)_Y := (S^{1/2}u, S^{1/2}v)$, embedded continuously and densely in X .

Let $B(Y, X)$ be the space of all bounded linear operators on a Banach space Y to another X , with norm $\|\cdot\|_{Y \rightarrow X}$. We shall also use the following abbreviation. Namely, $B(X) := B(X, X)$ and $B(Y) := B(Y, Y)$. We use the subscript $*$ to refer the strong operator topology in $B(Y, X)$. For instance, $F(\cdot) \in L^p_*(0, T, B(Y, X))$ for $1 \leq p \leq \infty$ means that $F(t) \in B(Y, X)$ is defined for a.a. $t \in (0, T)$, is strongly measurable, and there exists $\gamma_F \in L^p(0, T)$ such that $\|F(t)\|_{Y \rightarrow X} \leq \gamma_F(t)$ for a.a. $t \in (0, T)$ (for this notation see Kato [8] and Tanaka [16]).

The first purpose of this paper is to prove

Theorem 1.1. *Let $\{A(t)\}$ be a family of closed linear operators in a separable Hilbert space X , S a selfadjoint operator in X , satisfying (1.1). Assume that $A(t)$ satisfies following four conditions.*

(I) *There exists $\alpha \in L^1(0, T)$, $\alpha \geq 0$, such that*

$$(1.2) \quad |\operatorname{Re}(A(t)v, v)| \leq \alpha(t) \|v\|^2, \quad v \in D(A(t)), \text{ a.a. } t \in (0, T).$$

(II) *$Y = D(S^{1/2}) \subset D(A(t))$, a.a. $t \in (0, T)$.*

(III) *There exists $\beta \in L^1(0, T)$, $\beta \geq \alpha$, such that*

$$(1.3) \quad |\operatorname{Re}(A(t)u, Su)| \leq \beta(t) \|S^{1/2}u\|^2, \quad u \in D(S), \text{ a.a. } t \in (0, T).$$

(IV) *$A(\cdot) \in L^1_\star(0, T; B(Y, X))$, i.e., there exists $\gamma \in L^1(0, T)$ such that*

$$(1.4) \quad \|A(t)\|_{Y \rightarrow X} \leq \gamma(t), \quad \text{a.a. } t \in (0, T).$$

Then there exists a unique evolution operator $\{U(t, s); (t, s) \in \Delta\}$, where $\Delta := \{(t, s); 0 \leq s \leq t \leq T\}$, having the following properties.

(i) *$U(\cdot, \cdot)$ is strongly continuous on Δ to $B(X)$, with*

$$(1.5) \quad \|U(t, s)\|_{B(X)} \leq \exp\left(\int_s^t \alpha(r) dr\right), \quad (t, s) \in \Delta.$$

(ii) *$U(t, r)U(r, s) = U(t, s)$ on Δ and $U(s, s) = 1$ (the identity).*

(iii) *$U(t, s)Y \subset Y$ and $U(\cdot, \cdot)$ is strongly continuous on Δ to $B(Y)$, with*

$$(1.6) \quad \|U(t, s)\|_{B(Y)} \leq \exp\left(\int_s^t \beta(r) dr\right), \quad (t, s) \in \Delta.$$

Furthermore, let $v \in Y$, Then $U(\cdot, \cdot)v \in W^{1,1}(\Delta; X)$, with

(iv) *$(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v$, $(t, s) \in \Delta$, a.a. $t \in (s, T)$, and*

(v) *$(\partial/\partial s)U(t, s)v = U(t, s)A(s)v$, $(t, s) \in \Delta$, a.a. $s \in (0, t)$.*

In particular, if $A(\cdot) \in C([0, T]; B(Y, X))$, then Theorem 1.1 has already been proved in Mori [9] (unpublished). For lack of the continuity to the contrary we cannot approximate the family $\{A(\cdot)\}$ by a sequence $\{A_n(\cdot)\}$ of piecewise constant families. Therefore, we should consider some other approximation (see Definition 2.2 below).

Here we note that (III) is a consequence of conditions (I), (II) and the commutator type condition

(K) *There exists $B(\cdot) \in L^1_\star(0, T; B(X))$ such that*

$$S^{1/2}A(t)S^{-1/2} = A(t) + B(t), \quad \text{a.a. } t \in (0, T),$$

in which the domain relation is exact. Under condition (K) and the so-called stability condition, a similar theorem as in Theorem 1.1 was first established by Kato [4] and [5].

Under conditions (I)–(III) with $t = t_0$ fixed both $\alpha(t_0) \pm A(t_0)$ become m -accretive in X (see Lemma 2.1). Thus $A(t_0)$ together with $-A(t_0)$ is not in general the negative generator of an analytic C_0 -semigroup on X . That is, (E) is definitely an equation of hyperbolic type. In other words, “hyperbolic” may be replaced with “non-parabolic”.

In order to state the main theorem we need the notion of a strong solution. We say that $u(\cdot)$ is a *strong* solution of (E) if

- (i) $u(\cdot) \in W^{1,1}(0, T; X)$,
- (ii) $u(t) \in Y$ ($0 \leq t \leq T$), and
- (iii) $u(\cdot)$ satisfies (E) almost everywhere.

Note that $A(t)u(t)$ is meaningful. Under this definition we have

Theorem 1.2. *Let $u_0 \in Y$ and $f(\cdot) \in L^1(0, T; Y)$. If $u(\cdot)$ is defined by*

$$u(t) := U(t, 0)u_0 + \int_0^t U(t, s)f(s) ds,$$

then $u(\cdot) \in W^{1,1}(0, T; X) \cap C([0, T]; Y)$ and $u(\cdot)$ is a unique strong solution of (E) with $u(0) = u_0$.

In Section 2 we prepare some lemmas. Then we shall prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. In Section 5 we show the selfadjointness of some operators for applications. Last, in Section 6 we apply Theorem 1.1 to the Dirac equation.

2. Preliminaries

Let X be a separable Hilbert space.

Lemma 2.1. *Let A be a closed linear operator in X , satisfying*

$$\operatorname{Re}(Av, v) \geq -\alpha \|v\|^2, \quad v \in D(A),$$

where $\alpha \geq 0$ is a constant. Let S be a selfadjoint operator in X , with $D(S) \subset D(A)$, satisfying (1.1). Assume that there exist nonnegative constants β and γ such that for all $u \in D(S)$,

$$\operatorname{Re}(Au, Su) \geq -\gamma \|u\|^2 - \beta \|u\| \cdot \|Su\|.$$

Then

- (a) $A + \alpha$ is m -accretive in X .
- (b) $D(S)$ is a core for A .

This lemma was obtained by Kato [6]. For a complete proof see Okazawa [11].

Definition 2.2 (Ishii [3]). Let $\{A(t)\}$ be a family as above, satisfying (1.2)–(1.4). Put

$$A_n(t) := A(t) \left(1 + \frac{1}{\nu_n(t)} A(t) \right)^{-1} = \nu_n(t) \left[1 - \left(1 + \frac{1}{\nu_n(t)} A(t) \right)^{-1} \right],$$

$$\nu_n(t) := n(1 + \gamma(t)) + 2\beta(t), \quad n \in \mathbb{N}, \quad \text{a.a. } t \in (0, T).$$

Then $\{A_n(t)\}_{n \in \mathbb{N}}$ is called a *modified Yosida approximation* of $\{A(t)\}$.

If $t_0 \in (0, T)$ is fixed, then $\alpha(t_0)$, $\beta(t_0)$, $\gamma(t_0)$ and $A_n(t_0)$ are considered as nonnegative constants α , β , γ and the usual Yosida approximation of $A(t_0)$ (provided $\nu_n(t_0) > 2\beta$), respectively. Therefore the following lemmas are proved in the same way as in [12].

Lemma 2.3. *Let $A(t)$ be as in Definition 2.2. Then*

$$(a) \left\| \left(1 + \frac{1}{\nu_n(t)} A(t) \right)^{-1} \right\|_{B(X)} \leq \left(1 - \frac{\alpha(t)}{\nu_n(t)} \right)^{-1}, \quad n \in \mathbb{N}, \quad \text{a.a. } t \in (0, T).$$

$$(b) \operatorname{Re}(A_n(t)w, w) \geq -\alpha(t) \left(1 - \frac{\alpha(t)}{\nu_n(t)} \right)^{-1} \|w\|^2, \quad w \in X, \quad \text{a.a. } t \in (0, T).$$

$$(c) \|A_n(t)\|_{B(X)} \leq \nu_n(t), \quad n \in \mathbb{N}, \quad \text{a.a. } t \in (0, T).$$

Lemma 2.4. *Let $A(t)$ be as in Lemma 2.3. Assume that there exist $\beta \in L^1(0, T)$ and $\gamma \in \mathbb{R}$ such that $\beta \geq \alpha \geq 0$ and*

$$(2.1) \quad \operatorname{Re}(A(t)u, Su) \geq -\gamma \|u\|^2 - \beta(t)(u, Su) \quad \forall u \in D(S), \quad \text{a.a. } t \in (0, T),$$

where S is a selfadjoint operator in X satisfying (1.1). Then, for $S_\varepsilon := S(1 + \varepsilon S)^{-1}$,

$$\operatorname{Re}(A(t)u, S_\varepsilon u) \geq -\gamma \|u\|^2 - \beta(t)(u, S_\varepsilon u) \quad \forall u \in D(A(t)), \quad \text{a.a. } t \in (0, T).$$

Lemma 2.5. *Let $A(\cdot)$ and S be as in Lemma 2.4. Assume that (2.1) with $\gamma = 0$ is satisfied. Then*

$$(a) \left(1 + \frac{1}{\nu_n(t)} A(t) \right)^{-1} D(S^{1/2}) \subset D(S^{1/2}), \quad \text{a.a. } t \in (0, T), \text{ with}$$

$$\left\| S^{1/2} \left(1 + \frac{1}{\nu_n(t)} A(t) \right)^{-1} v \right\| \leq \left(1 - \frac{\beta(t)}{\nu_n(t)} \right)^{-1} \|S^{1/2} v\|, \quad v \in D(S^{1/2}), \quad \text{a.a. } t \in (0, T).$$

$$(b) \operatorname{Re}(A_n(t)w, S_\varepsilon w) \geq -\beta(t) \left(1 - \frac{\beta(t)}{\nu_n(t)} \right)^{-1} (w, S_\varepsilon w), \quad w \in X, \quad \text{a.a. } t \in (0, T).$$

Lemma 2.6. *Let $\{A_n\}$ be the Yosida approximation of a linear m -accretive operator A in X . Let $\{w_n\}$ be a sequence in X such that $w_n \rightarrow u$ ($n \rightarrow \infty$) weakly in X . If $\{A_n w_n\}$ is bounded, then $u \in D(A)$ and $A_n w_n \rightarrow Au$ ($n \rightarrow \infty$) weakly in X .*

3. Construction of evolution operators

In this section we shall prove Theorem 1.1. Let $\{A(t)\}$ be a family of closed linear operators in a separable Hilbert space X . Let S be a selfadjoint operator in X , satisfying (1.1). Since we need conditions (I) and (III) as a whole only in the last step of the proof (see Lemmas 3.9 and 3.11 below), we may introduce weaker conditions $(I)_+$ and $(III)_+$. Namely assume that

$(I)_+$ There exists $\alpha \in L^1(0, T)$, $\alpha \geq 0$ such that

$$\operatorname{Re}(A(t)v, v) \geq -\alpha(t) \|v\|^2, \quad v \in D(A(t)), \quad \text{a.a. } t \in (0, T).$$

(II) $Y = D(S^{1/2}) \subset D(A(t))$, a.a. $t \in (0, T)$.

(III)₊ There exists $\beta \in L^1(0, T)$, $\beta \geq \alpha$ such that

$$\operatorname{Re}(A(t)u, Su) \geq -\beta(t) \|S^{1/2}u\|^2, \quad u \in D(S), \text{ a.a. } t \in (0, T).$$

(IV) $A(\cdot) \in L_*^1(0, T; B(Y, X))$ with $\|A(t)\|_{Y \rightarrow X} \leq \gamma(t)$, a.a. $t \in (0, T)$.

Under these conditions we shall construct a two parameter family $\{U(t, s); (t, s) \in \Delta\}$ in $B(X)$, satisfying among others (i), (ii), (iv) and (v) of Theorem 1.1.

First of all, by virtue of conditions (I)₊, (II) and (III)₊ we see from Lemma 2.1 (a) that $A(t) + \alpha(t)$ is m -accretive in X for almost all $t \in (0, T)$.

Lemma 3.1. *Let $\{A_n(t)\}$ and $\{\nu_n(t)\}$ be as in Definition 2.2. Then*

(a) $A_n(\cdot) \in L_*^1(0, T; B(X))$ with $\|A_n(t)\|_{B(X)} \leq \nu_n(t)$, a.a. $t \in (0, T)$.

(b) $\|A(t)v - A_n(t)v\| \rightarrow 0$, $\forall v \in D(A(t))$, a.a. $t \in (0, T)$.

Proof. (a) follows from Lemma 2.3 (c).

(b) is well-known as a property of the Yosida approximation. □

Proposition 3.2. *Let $s \in [0, T)$. Then the approximate problem:*

$$(3.1) \quad \begin{cases} (d/dt)u_n(t) + A_n(t)u_n(t) = 0, & \text{a.a. } t \in (s, T), \\ u_n(s) = w \end{cases}$$

has a unique strong solution $u_n \in W^{1,1}(s, T; X)$.

In particular, if $A_n(\cdot) \in C([0, T]; B(Y, X))$, then the assertion is found in Pazy [15, Section 5.1]. The proof is standard (see e.g. Brézis [1, Theorem VII.3]).

We define the “solution operator” of the approximate problem by

$$U_n(t, s)w := u_n(t) \quad \text{for } (t, s) \in \Delta$$

where u_n is the solution of (3.1). The main properties of $U_n(t, s)$ are given in the next lemma (cf. [15, Section 5.1]).

Lemma 3.3. *For every $n \in \mathbb{N}$, let $\{A_n(t)\}$ and $\{U_n(t, s)\}$ be as defined above. Then $\{U_n(t, s)\}$ is a sequence of bounded linear operators on X , with*

(a) $\|U_n(t, s)\|_{B(X)} \leq \exp\left(\int_s^t \nu_n(r) dr\right)$ on Δ .

(b) $U_n(t, r)U_n(r, s) = U_n(t, s)$ on Δ and $U_n(s, s) = 1$.

(c) $U_n(\cdot, \cdot)$ is uniformly continuous on Δ .

(d) $(\partial/\partial t)U_n(t, s)w = -A_n(t)U_n(t, s)w$, $w \in X$, $(t, s) \in \Delta$, a.a. $t \in (s, T)$.

(e) $(\partial/\partial s)U_n(t, s)w = U_n(t, s)A_n(s)w$, $w \in X$, $(t, s) \in \Delta$, a.a. $s \in (0, t)$.

For the limiting procedure we need the following

Lemma 3.4. *Let $\{U_n(t, s)\}$ and $\nu_n(t)$ be as in Lemma 3.3. Then*

$$(a) \|U_n(t, s)\|_{B(X)} \leq \exp\left[\int_s^t \alpha(r) \left(1 - \frac{\alpha(r)}{\nu_n(r)}\right)^{-1} dr\right] \leq \exp\left(2 \int_s^t \alpha(r) dr\right) \text{ on } \Delta.$$

(b) $U_n(t, s)Y \subset Y$ and

$$\|U_n(t, s)\|_{B(Y)} \leq \exp\left[\int_s^t \beta(r) \left(1 - \frac{\beta(r)}{\nu_n(r)}\right)^{-1} dr\right] \leq \exp\left(2 \int_s^t \beta(r) dr\right) \text{ on } \Delta.$$

(c) For $v \in Y$, $\|A_n(t)U_n(t, s)v\| \leq 2\gamma(t) \exp\left(2 \int_s^t \beta(r) dr\right) \|v\|_Y$, a.a. $(t, s) \in \Delta$.

Proof. First we prove (b). Let $\{S_\varepsilon\}$ be the Yosida approximation of S . Since S_ε is a bounded linear operator on X , we see from Lemma 3.3 (d) and Lemma 2.5 (b) that for $v \in Y$, a.a. $r \in (s, T)$,

$$(3.2) \quad \begin{aligned} (\partial/\partial r) \|S_\varepsilon^{1/2}U_n(r, s)v\|^2 &= -2 \operatorname{Re}(A_n(r)U_n(r, s)v, S_\varepsilon U_n(r, s)v) \\ &\leq 2\beta(r) \left(1 - \frac{\beta(r)}{\nu_n(r)}\right)^{-1} \|S_\varepsilon^{1/2}U_n(r, s)v\|^2. \end{aligned}$$

Integrating this inequality on $[s, t]$. By the Gronwall inequality we have

$$\begin{aligned} \|S_\varepsilon^{1/2}U_n(r, s)v\|^2 &\leq \exp\left[2 \int_s^t \beta(r) \left(1 - \frac{\beta(r)}{\nu_n(r)}\right)^{-1} dr\right] \|S_\varepsilon^{1/2}v\|^2 \\ &\leq \exp\left[2 \int_s^t \beta(r) \left(1 - \frac{\beta(r)}{\nu_n(r)}\right)^{-1} dr\right] \|S^{1/2}v\|^2. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we can obtain the first inequality of (b). The second inequality is trivial because $\nu_n(t) \geq 2\beta(t)$ a.a. $t \in (0, T)$.

(a) is proved similarly by Lemma 2.3 (b), starting with

$$(\partial/\partial r) \|U_n(r, s)w\|^2 = -2 \operatorname{Re}(A_n(r)U_n(r, s)w, U_n(r, s)w).$$

(c) follows from (b). In fact, we see from conditions (II), (IV) and Lemma 2.3 (a) that

$$(3.3) \quad \|A_n(t)v\| \leq \left(1 - \frac{\alpha(t)}{\nu_n(t)}\right)^{-1} \|A(t)v\| \leq 2\gamma(t) \|v\|_Y, \quad \text{a.a. } t \in (0, T).$$

The assertion follows from (b). □

Lemma 3.5. *Let $\{U_n(t, s)\}$ be as in Lemma 3.3. Then there is a family $\{U(t, s); (t, s) \in \Delta\}$ in $B(X)$ such that*

(a) $U(t, s) := s\text{-}\lim_{n \rightarrow \infty} U_n(t, s)$, where the convergence is uniform on Δ , and hence $U(\cdot, \cdot)$ is strongly continuous on Δ to $B(X)$, with

$$(3.4) \quad \|U(t, s)v - U_n(t, s)v\|^2 \leq \frac{2}{n} \|\gamma\|_{L^1(s, t)} \exp\left(4 \int_s^t \beta(r) dr\right) \|v\|_Y^2, \quad v \in Y$$

and $\|U(t, s)\|_{B(X)} \leq \exp\left(\int_s^t \alpha(r) dr\right)$ on Δ .

(b) $U(t, r)U(r, s) = U(t, s)$ on Δ and $U(s, s) = 1$.

(c) $U(t, s)Y \subset Y$ and $S^{1/2}U(t, s)v = \text{w-lim}_{n \rightarrow \infty} S^{1/2}U_n(t, s)v$, with

$$(3.5) \quad \|S^{1/2}U(t, s)v\| \leq \exp\left(\int_s^t \beta(r) dr\right) \|S^{1/2}v\|, \quad v \in Y, \quad (t, s) \in \Delta.$$

Proof. (a) Let $v \in Y$. Then we shall show that

$$(3.6) \quad \|U_n(t, s)v - U_m(t, s)v\|^2 \leq 2 \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}} \right|^2 \|\gamma\|_{L^1(s, t)} \exp\left(4 \int_s^t \beta(r) dr\right) \|v\|_Y^2.$$

The computation is similar as in [12]. Put

$$\begin{aligned} u_{nm}(r, s) &:= U_n(r, s)v - U_m(r, s)v, \\ w_{nm}(r, s) &:= J_n(r)U_n(r, s)v - J_m(r)U_m(r, s)v, \end{aligned}$$

where $J_n(r) := (1 + \nu_n(r)^{-1}A(r))^{-1} = 1 - \nu_n(r)^{-1}A_n(r)$. Then by Lemma 3.3 (d) we have

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial r} \|u_{nm}(r, s)\|^2 \\ &= -\text{Re}(A_n(r)U_n(r, s)v - A_m(r)U_m(r, s)v, u_{nm}(r, s) - w_{nm}(r, s)) \\ & \quad - \text{Re}(A(r)w_{nm}(r, s), w_{nm}(r, s)). \end{aligned}$$

Noting that

$$(3.7) \quad u_{nm}(r, s) - w_{nm}(r, s) = \nu_n(r)^{-1}A_n(r)U_n(r, s)v - \nu_m(r)^{-1}A_m(r)U_m(r, s)v,$$

we see that

$$\begin{aligned} & -\text{Re}(A_n(r)U_n(r, s)v - A_m(r)U_m(r, s)v, u_{nm}(r, s) - w_{nm}(r, s)) \\ &= (\nu_n(r)^{-1} + \nu_m(r)^{-1}) \text{Re}(A_n(r)U_n(r, s)v, A_m(r)U_m(r, s)v) \\ & \quad - \nu_n(r)^{-1} \|A_n(r)U_n(r, s)v\|^2 - \nu_m(r)^{-1} \|A_m(r)U_m(r, s)v\|^2. \end{aligned}$$

On the other hand, it follows from condition (I)₊ that

$$\begin{aligned} -\text{Re}(A(r)w_{nm}(r, s), w_{nm}(r, s)) &\leq \alpha(r) \|w_{nm}(r, s)\|^2 \\ &\leq \beta(r) \|w_{nm}(r, s)\|^2. \end{aligned}$$

We see from (3.7) that $\|w_{nm}(r, s)\|^2$ is estimated as follows:

$$\begin{aligned} & \frac{1}{2} \|w_{nm}(r, s)\|^2 - \|u_{nm}(r, s)\|^2 \\ &\leq \|\nu_n(r)^{-1}A_n(r)U_n(r, s)v - \nu_m(r)^{-1}A_m(r)U_m(r, s)v\|^2 \\ &= \nu_n(r)^{-2} \|A_n(r)U_n(r, s)v\|^2 + \nu_m(r)^{-2} \|A_m(r)U_m(r, s)v\|^2 \\ & \quad - 2\nu_n(r)^{-1}\nu_m(r)^{-1} \text{Re}(A_n(r)U_n(r, s)v, A_m(r)U_m(r, s)v). \end{aligned}$$

Combining these estimates and using Lemma 3.4 (c), we have

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial r} \|u_{nm}(r, s)\|^2 - 2\beta(r) \|u_{nm}(r, s)\|^2 \\ & \leq \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}} \right|^2 \gamma(r) \exp\left(4 \int_s^r \beta(\tau) d\tau\right) \|v\|_Y^2. \end{aligned}$$

Integrating this inequality on $[s, t]$, we obtain (3.6). Since Y is dense in X , we see from Lemma 3.4 (a) that the family $\{U(t, s); (t, s) \in \Delta\}$ in $B(X)$ is defined: for $w \in X$,

$$U_n(\cdot, \cdot)w \rightarrow U(\cdot, \cdot)w \quad \text{in } C(\Delta; X) \quad \text{as } n \rightarrow \infty.$$

(b) follows from Lemma 3.3 (b).

(c) is a consequence of (a) and Lemma 3.4 (b). □

Lemma 3.6. *Let $\{U(t, s)\}$ be as in Lemma 3.5. Let $v \in Y$ and $(t, s) \in \Delta$. Then*

(a) $U(t, s)v \in D(A(t))$, and

$$\|A(t)U(t, s)v\| \leq \gamma(t) \exp\left[\int_s^t \beta(r) dr\right] \|v\|_Y \quad \text{a.a. } t \in (s, T)$$

with

$$(3.8) \quad A(t)U(t, s)v = \text{w-lim}_{n \rightarrow \infty} A_n(t)U_n(t, s)v \quad \text{a.a. } t \in (s, T).$$

$$(b) \quad \int_s^t U(t, r)A(r)v dr = \text{s-lim}_{n \rightarrow \infty} \int_s^t U_n(t, r)A_n(r)v dr \quad \text{in } X.$$

$$(c) \quad (\partial/\partial s)U(t, s)v = U(t, s)A(s)v \quad \text{a.a. } s \in (0, t).$$

Proof. (a) $A(\cdot)U(\cdot, s)v \in L^1(s, t; X)$ follows from condition (IV) and (3.5). By virtue of Lemma 2.6, (3.8) follows from Lemmas 3.4 (c) and 3.5 (c).

(b) For a.a. $r \in (s, t)$, it follows from Lemmas 3.1 (b), 3.4 (a) and 3.5 (a) that

$$U(t, r)A(r)v = \text{s-lim}_{n \rightarrow \infty} U_n(t, r)A_n(r)v \quad \text{in } X.$$

On the other hand, Lemma 3.4 (a) and (3.3) yield that

$$\|U_n(t, r)A_n(r)v\| \leq 2\gamma(r) \exp\left(2 \int_0^T \alpha(\tau) d\tau\right) \|v\|_Y \in L^1(s, t).$$

Therefore we obtain the assertion by the Lebesgue convergence theorem.

(c) By Lemma 3.3 (e) we have

$$v - U_n(t, s)v = \int_s^t U_n(t, r)A_n(r)v dr, \quad v \in Y.$$

Letting $n \rightarrow \infty$, we see from (3.4) and (b) that

$$(3.9) \quad v - U(t, s)v = \int_s^t U(t, r)A(r)v dr, \quad v \in Y.$$

Since condition (IV) and Lemma 3.5 (a), $U(t, \cdot)A(\cdot)v \in L^1(0, t; X)$. Therefore (3.9) is strongly differentiable on a.a. $s \in (0, t)$ and we obtain the assertion. □

Lemma 3.7. *Let $\{U(t, s)\}$ be as in Lemma 3.5. Let $v \in Y$. Then*

(a) *For each $s \in [0, T]$, $A(\cdot)U(\cdot, s)v$ is Bochner integrable on $[s, T]$, with*

$$(3.10) \quad U(t, s)v = v - \int_s^t A(r)U(r, s)v \, dr, \quad t \in [s, T],$$

and hence $U(\cdot, s)$ is absolutely continuous on $[s, T]$:

$$(3.11) \quad \|U(t, s)v - U(t', s)v\| \leq \left| \int_{t'}^t \gamma(r) \, dr \right| \exp \left[\int_0^T \beta(r) \, dr \right] \|v\|_Y.$$

(b) $(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v$, a.a. $t \in (s, T)$.

Proof. (a) It follows from Lemma 3.6 (a) that $A(\cdot)U(\cdot, s)v$ is Bochner integrable on $[s, T]$. Now Lemma 3.3 (d) implies that for each $w \in X$,

$$(U_n(t, s)v, w) = (v, w) - \int_s^t (A_n(r)U_n(r, s)v, w) \, dr.$$

Letting $n \rightarrow \infty$, we see from (3.4) and (3.8) that

$$(U(t, s)v, w) = (v, w) - \int_s^t (A(r)U(r, s)v, w) \, dr.$$

Thus we obtain (3.10) and (3.11).

(b) is a direct consequence of (3.10). □

It is easy to prove the uniqueness of the evolution operator constructed above.

Lemma 3.8. *Let $\{U(t, s)\}$ be as in Lemma 3.5. Suppose that $\{V(t, s)\}$ is another family in $B(X)$ with the properties (i), (ii) and (v). Then $U(t, s) \equiv V(t, s)$ on Δ .*

In fact, we see from Lemma 3.7 (b) that for $v \in Y$,

$$(\partial/\partial r)V(t, r)U(r, s)v = 0 \quad \text{a.a. } r \in (s, t).$$

Hence we obtain $U(t, s)v = V(t, s)v$. Since Y is dense in X , the assertion follows.

Lemma 3.9. *Let $\{A(t)\}$ and S be as in Theorem 1.1. Assume that conditions (I) and (III) are satisfied, with the inclusion $D(S) \subset D(A(t))$. Let $\{S_\epsilon\}$ be the Yosida approximation of S . Then*

$$|\operatorname{Re}(A(t)v, S_\epsilon v)| \leq \beta(t)(v, S_\epsilon v), \quad v \in D(A(t)), \quad \text{a.a. } t \in (0, T).$$

In particular, if $D(S^{1/2}) \subset D(A(t))$ (this is condition (II)), then

$$(3.12) \quad |\operatorname{Re}(A(t)v, S_\epsilon v)| \leq \beta(t) \|S^{1/2}v\|^2, \quad v \in D(S^{1/2}), \quad \text{a.a. } t \in (0, T).$$

The conclusion follows from Lemma 2.4 (this fact is first noted in [13]).

Lemma 3.10. *Let $\{U(t, s)\}$ be as in Lemma 3.5. Let $v \in Y$. Then*

- (a) $S^{1/2}U(t, s)v$ is weakly continuous on Δ .
- (a') $S^{1/4}U(t, s)v$ is strongly continuous on Δ .
- (b) $S^{1/2}U(t, s)v \rightarrow S^{1/2}v$ as $(t, s) \rightarrow (t_0, t_0)$.
- (c) For $t \in (0, T]$, $U(t, \cdot)v \in C([0, T]; Y)$.

Proof. (a) Let $\{S_\varepsilon\}$ be the Yosida approximation of S . Then for $v \in Y$, $S_\varepsilon^{1/2}U(t, s)v$ is continuous on Δ . Noting that $(1 + \varepsilon S)^{-1/2}w \rightarrow w$ ($\varepsilon \downarrow 0$), we see by (3.5) that

$$S^{1/2}U(t, s)v = \text{w-lim}_{\varepsilon \downarrow 0} S_\varepsilon^{1/2}U(t, s)v,$$

where the convergence is uniform on Δ and hence the limit function is also weakly continuous on Δ .

- (a') is a direct consequence of Lemma 3.5 (a) and (3.5).
- (b) Let $t_0 \in [0, T]$. Then it suffices by (a) to show that

$$\|S^{1/2}U(t, s)v\| \rightarrow \|S^{1/2}v\| \quad \text{as } (t, s) \rightarrow (t_0, t_0).$$

We see again by (a) that

$$\|S^{1/2}v\| \leq \liminf_{(t,s) \rightarrow (t_0,t_0)} \|S^{1/2}U(t, s)v\|.$$

On the other hand, it follows from (3.5) that

$$\limsup_{(t,s) \rightarrow (t_0,t_0)} \|S^{1/2}U(t, s)v\| \leq \|S^{1/2}v\|.$$

- (c) follows from (b) and (3.5). □

Now we are in a position to prove (iii) and $U(\cdot, \cdot) \in W^{1,1}(\Delta; B(Y, X))$ of Theorem 1.1.

Lemma 3.11. *Let $\{A(t)\}$ and S be as in Theorem 1.1. Assume that conditions (I)–(IV) are satisfied. Let $\{U(\cdot, \cdot)\}$ be as in Lemma 3.5. Then*

- (a) For $v \in Y$ and $s \in [0, T]$, $U(\cdot, s)v \in C([s, T]; Y)$.
- (b) $U(\cdot, \cdot)$ is strongly continuous on Δ to $B(Y)$.
- (c) For $v \in Y$, $U(\cdot, \cdot)v \in W^{1,1}(\Delta; X)$.

Proof. (a) Lemmas 3.5 (a) and 3.7 (b) yield that $U(\cdot, s)v \in W^{1,1}(s, T; X) \subset C([s, T]; X)$. Thus it suffices to show that

$$(3.13) \quad S^{1/2}U(\cdot, s)v \in C([s, T]; X).$$

Let $t_0 \in [s, T]$. Then we have

$$\begin{aligned} \|S^{1/2}U(t, s)v - S^{1/2}U(t_0, s)v\|^2 &= \|S^{1/2}U(t, s)v\|^2 - \|S^{1/2}U(t_0, s)v\|^2 \\ &\quad - 2\operatorname{Re}(S^{1/2}U(t, s)v - S^{1/2}U(t_0, s)v, S^{1/2}U(t_0, s)v). \end{aligned}$$

Since $S^{1/2}U(t, s)v$ is weakly continuous on Δ (see Lemma 3.10 (a)), we obtain (3.13) if we show that

$$(3.14) \quad \|S^{1/2}U(t, s)v\|^2 \rightarrow \|S^{1/2}U(t_0, s)v\|^2 \quad \text{as } t \rightarrow t_0.$$

To this end we can use (3.2). Integrating (3.2) on $[t_0, t]$, we have

$$\|S_\varepsilon^{1/2}U_n(t, s)v\|^2 - \|S_\varepsilon^{1/2}U_n(t_0, s)v\|^2 = -2 \int_{t_0}^t \operatorname{Re}(A_n(r)U_n(r, s)v, S_\varepsilon U_n(r, s)v) dr.$$

Letting $n \rightarrow \infty$, we see from (3.4), (3.8) and Lemma 3.4 (c) that

$$\|S_\varepsilon^{1/2}U(t, s)v\|^2 - \|S_\varepsilon^{1/2}U(t_0, s)v\|^2 = -2 \int_{t_0}^t \operatorname{Re}(A(r)U(r, s)v, S_\varepsilon U(r, s)v) dr.$$

It follows from (3.12) and (3.5) that

$$\begin{aligned} \left| \|S_\varepsilon^{1/2}U(t, s)v\|^2 - \|S_\varepsilon^{1/2}U(t_0, s)v\|^2 \right| &\leq 2 \left| \int_{t_0}^t \beta(r) \exp \left[2 \int_s^r \beta(\tau) d\tau \right] dr \right| \|v\|_Y^2 \\ &= \left| \exp \left[2 \int_s^t \beta(r) dr \right] - \exp \left[2 \int_s^{t_0} \beta(r) dr \right] \right| \|v\|_Y^2. \end{aligned}$$

Noting that $(1 + \varepsilon S)^{-1}w \rightarrow w$ ($\varepsilon \downarrow 0$) for every $w \in X$, we have

$$\left| \|S^{1/2}U(t, s)v\|^2 - \|S^{1/2}U(t_0, s)v\|^2 \right| \leq \left| \exp \left[2 \int_s^t \beta(r) dr \right] - \exp \left[2 \int_s^{t_0} \beta(r) dr \right] \right| \|v\|_Y^2.$$

Thus we obtain (3.14).

(b) We follow the idea in Kato [4, Remark 5.4]. First let $t_0 = s_0$. Then the assertion follows from Lemma 3.10 (b). Next let $s_0 < t_0$. Set $a := 2^{-1}(s_0 + t_0)$. Then $s < a < t$ for $(t, s) \in B((t_0, s_0), 2^{-1}(t_0 - s_0)) \cap \Delta$. Thus we have

$$\begin{aligned} \|U(t, s)v - U(t_0, s_0)v\|_Y &\leq \|U(t, a)\|_{B(Y)} \|U(a, s)v - U(a, s_0)v\|_Y + \|(U(t, a) - U(t_0, a))U(a, s_0)v\|_Y. \end{aligned}$$

Therefore the assertion follows from (a), (3.5) and Lemma 3.10 (c).

(c) $U(\cdot, \cdot)v \in C(\Delta; X)$ is a direct consequence of (b). It follows from Lemma 3.5 (c) and 3.7 (b) that

$$\begin{aligned} \iint_{\Delta} \|(\partial/\partial t)U(t, s)v\| dt ds &= \iint_{\Delta} \|A(t)U(t, s)v\| dt ds \\ &\leq \iint_{\Delta} \gamma(t) \exp \left[\int_s^t \beta(r) dr \right] \|v\|_Y dt ds \\ &\leq T \|\gamma\|_{L^1(0, T)} \exp \left[\int_0^T \beta(r) dr \right] \|v\|_Y. \end{aligned}$$

Similarly by Lemma 3.5 (a) and 3.6 (c) we have

$$\iint_{\Delta} \|(\partial/\partial s)U(t, s)v\| \, dt ds \leq T \|\gamma\|_{L^1(0, T)} \exp\left[\int_0^T \alpha(r) \, dr\right] \|v\|_Y.$$

Therefore the assertion follows. \square

4. Inhomogeneous equations

In this section we prove Theorem 1.2. Let $A(t)$ and S be as in Theorem 1.1. First assume that condition (I)₊, (II), (III)₊ and (IV) are satisfied. Let $\{U(t, s); (t, s) \in \Delta\}$ be the evolution operator with the properties stated in Lemmas 3.5–3.7. Then for $u_0 \in Y$,

$$(4.1) \quad (d/dt)U(t, 0)u_0 + A(t)U(t, 0)u_0 = 0 \quad \text{a.a. } t \in (0, T).$$

Let $f(\cdot) \in L^1(0, T; Y)$ and put

$$(4.2) \quad v(t) := \int_0^t U(t, s)f(s) \, ds.$$

Then clearly $v(\cdot) \in L^\infty(0, T; X)$. We want to show that

$$(4.3) \quad (d/dt)v(t) + A(t)v(t) = f(t) \quad \text{a.a. } t \in (0, T).$$

Lemma 4.1. *Let $v(\cdot)$ be as above and $t \in [0, T]$. Then*

$$(a) \, v(\cdot) \in L^\infty(0, T; Y), \text{ with } \|v(t)\|_Y \leq \exp\left[\int_0^T \beta(r) \, dr\right] \|f(\cdot)\|_{L^1(0, T; Y)}.$$

$$(b) \, S^{1/2}v(\cdot) \text{ is weakly continuous on } [0, T].$$

$$(c) \, v(t) \in D(A(t)) \text{ and } \|A(\cdot)v(\cdot)\|_{L^1(0, T; X)} \leq \|\gamma\|_{L^1(0, T)} \|v(\cdot)\|_{L^\infty(0, T; Y)}.$$

Proof. (a) Let $\{S_\epsilon\}$ be the Yosida approximation of S . Then we have

$$S_\epsilon^{1/2}v(t) = \int_0^t S_\epsilon^{1/2}U(t, s)f(s) \, ds.$$

Since $\|S_\epsilon^{1/2}w\| \leq \|S^{1/2}w\| \leq \|w\|_Y$, it follows from (3.5) that

$$\|S_\epsilon^{1/2}v(t)\| \leq \int_0^t \|U(t, s)\|_{B(Y)} \|f(s)\|_Y \, ds \leq \exp\left[\int_0^T \beta(r) \, dr\right] \|f(\cdot)\|_{L^1(0, T; Y)}$$

Hence we see that $v(t) \in Y$ and

$$(4.4) \quad S^{1/2}v(t) = \text{w-}\lim_{\epsilon \downarrow 0} S_\epsilon^{1/2}v(t), \quad t \in [0, T].$$

Thus the assertion follows.

(b) The convergence in (4.4) is uniform on $[0, T]$ and therefore $S^{1/2}v(\cdot)$ is weakly continuous on $[0, T]$.

(c) follows from (a) and the condition (II). \square

Next let $\{U_n(t, s)\}$ be as in Theorem 3.2 and put

$$v_n(t) := \int_0^t U_n(t, s) f(s) ds.$$

Then $v_n(\cdot) \in W^{1,1}(0, T; X)$ and

$$(4.5) \quad (d/dt)v_n(t) = -A_n(t)v_n(t) + f(t) \quad \text{a.a. } t \in (0, T).$$

Now we can prove (4.3).

Lemma 4.2. *Let $v(\cdot)$ be as above. Then*

- (a) $v_n(\cdot) \rightarrow v(\cdot)$ in $C([0, T]; X)$ as $n \rightarrow \infty$.
- (b) $A(t)v(t) = \lim_{n \rightarrow \infty} A_n(t)v_n(t) \quad \text{a.a. } t \in (0, T)$.
- (c) $A(\cdot)v(\cdot)$ is Bochner integrable on $[0, T]$ and

$$(4.6) \quad v(t) = - \int_0^t A(s)v(s) ds + \int_0^t f(s) ds.$$

- (d) $(d/dt)v(t) = -A(t)v(t) + f(t) \quad \text{a.a. } t \in (0, T)$.

Proof. (a) follows from (3.4).

(b) (a) and Lemma 4.1 (c) implies by Lemma 2.6 that $A(\cdot)v(\cdot)$ is the weak limit of $A_n(\cdot)v_n(\cdot)$ as $n \rightarrow \infty$.

(c) It follows from (b) that $A(\cdot)v(\cdot)$ is strongly measurable. Furthermore, by Lemma 4.1 (c) we have $A(\cdot)v(\cdot) \in L^1(0, T; X)$. Therefore $A(\cdot)v(\cdot)$ is Bochner integrable on $[0, T]$. On the other hand, we see from (4.5) that for each $w \in X$,

$$(v_n(t), w) = - \int_0^t (A_n(s)v_n(s), w) ds + \int_0^t (f(s), w) ds.$$

Letting $n \rightarrow \infty$, we have

$$(v(t), w) = - \int_0^t (A(s)v(s), w) ds + \int_0^t (f(s), w) ds.$$

Hence we obtain (4.6).

- (d) Strong differentiability of $v(t)$ is a consequence of (4.6). □

The next lemma guarantees that the strong solution of (E) is expressed by the variation of constant formula.

Lemma 4.3. *Let $\{U(t, s)\}$ be the evolution operator with properties (i), (ii) and (v). Let $u(\cdot)$ be a strong solution of (E) with $u(0) = u_0 \in Y$. If $f \in L^1(0, T; X)$ then*

$$(4.7) \quad u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s) ds.$$

In fact, it suffices to integrate the identity:

$$(\partial/\partial s)U(t, s)u(s) = U(t, s)f(s) \quad \text{a.a. } s \in (0, t).$$

Consequently, it follows from (4.1) and (4.3) that if $f(\cdot) \in L^1(0, T; Y)$ then $u(\cdot)$ given by (4.7) is a unique solution of (E) with $u(0) = u_0 \in Y$.

Now we are in a position to prove Theorem 1.2.

Lemma 4.4. *Let $\{A(t)\}$ and S be as in Theorem 1.1. Assume that conditions (I)–(IV) are satisfied. Let $\{U(t, s)\}$ be the evolution operator on X generated by $\{A(t)\}$. For $f(\cdot) \in L^1(0, T; Y)$ let $v(\cdot)$ be as in (4.2). Then*

$$v(\cdot) \in W^{1,1}(0, T; X) \cap C([0, T]; Y).$$

Proof. It follows from Lemma 4.2 (d) that $v \in W^{1,1}(0, T; X)$. Hence it suffices to show that

$$(4.8) \quad v(\cdot) \in C([0, T]; Y).$$

This is shown by the similar way as in Lemma 3.11 (a). Let $\{S_\varepsilon\}$ be the Yosida approximation of S . Then it follows from (4.5) that

$$\begin{aligned} (d/ds) \|S_\varepsilon^{1/2}v_n(s)\|^2 &= 2 \operatorname{Re}((d/ds)v_n(s), S_\varepsilon v_n(s)) \\ &= 2 \operatorname{Re}(-A_n(s)v_n(s) + f(s), S_\varepsilon v_n(s)) \quad \text{a.a. } s \in (0, T). \end{aligned}$$

Integrating this equality from $s = t_0$ to $s = t$, we have

$$\begin{aligned} &\|S_\varepsilon^{1/2}v_n(t)\|^2 - \|S_\varepsilon^{1/2}v_n(t_0)\|^2 \\ &= -2 \int_{t_0}^t \operatorname{Re}(A_n(s)v_n(s), S_\varepsilon v_n(s)) ds + 2 \int_{t_0}^t \operatorname{Re}(f(s), S_\varepsilon v_n(s)) ds. \end{aligned}$$

Letting $n \rightarrow \infty$, we see from Lemma 4.2 (a) and (b) that

$$\begin{aligned} &\|S_\varepsilon^{1/2}v(t)\|^2 - \|S_\varepsilon^{1/2}v(t_0)\|^2 \\ &= -2 \int_{t_0}^t \operatorname{Re}(A(s)v(s), S_\varepsilon v(s)) ds + 2 \int_{t_0}^t \operatorname{Re}(f(s), S_\varepsilon v(s)) ds. \end{aligned}$$

It follows from (3.12) and Lemma 4.1 (a) that

$$\begin{aligned} &\left| \|S_\varepsilon^{1/2}v(t)\|^2 - \|S_\varepsilon^{1/2}v(t_0)\|^2 \right| \\ &\leq 2 \left| \int_{t_0}^t \beta(t) \|S^{1/2}v(s)\|^2 ds \right| + 2 \left| \int_{t_0}^t \|S^{1/2}f(s)\| \cdot \|S^{1/2}v(s)\| ds \right| \\ &\leq 2 \|\beta\|_{L^1(t_0, t)} \|v(\cdot)\|_{L^\infty(0, T; Y)}^2 + 2 \|f(\cdot)\|_{L^1(t_0, t; Y)} \|v(\cdot)\|_{L^\infty(0, T; Y)}. \end{aligned}$$

Thus we have

$$(4.9) \quad \|S_\varepsilon^{1/2}v(t)\|^2 \rightarrow \|S_\varepsilon^{1/2}v(t_0)\|^2 \quad (t \rightarrow t_0).$$

By both Lemma 4.1 (b) and (4.9) we obtain (4.8). \square

In view of Lemma 4.2 (d) and Lemma 3.11 this completes the proof of Theorem 1.2.

5. Preliminaries for applications

Put $\langle x \rangle := (1 + |x|^2)^{1/2}$. In this section we consider the selfadjointness of

$$(5.1) \quad S := (H_D + V)^2 + \langle x \rangle^2 I \quad \text{for } u \in D(S) := \{u \in L^2(\mathbb{R}^3)^4; Su \in L^2(\mathbb{R}^3)^4\}.$$

Here H_D is the free Dirac operator

$$H_D := \alpha \cdot p + m\beta = \sum_{j=1}^3 \alpha_j i^{-1} \frac{\partial}{\partial x_j} + m\beta,$$

acting in the Hilbert space $L^2(\mathbb{R}^3)^4$; $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = \alpha_4$ are the usual 4×4 Hermitian matrices satisfying the commutation relations

$$(5.2) \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \quad (j, k = 1, 2, 3, 4),$$

and m is a positive constant (cf. Fattorini [2]).

The potential V is an operator of multiplication with a 4×4 Hermitian matrix-valued, measurable function $V(x)$ defined on \mathbb{R}^3 . It is assumed that

$$(5.3) \quad |V(x)| \leq a|x|^{-1} + b,$$

where $|V(x)|$ denotes the operator norm of $V(x) : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ and a, b are nonnegative constants with $a < 1/2$.

First, we consider the selfadjointness of $H_D + V$.

Theorem 5.1 (Kato-Rellich theorem). *Let A be a selfadjoint operator in a Hilbert space H and B a symmetric operator in H , with $D(A) \subset D(B)$. Assume that there exist two constants $a_0, b_0 \geq 0$ such that for all $u \in D(A)$,*

$$\|Bu\| \leq a_0\|u\| + b_0\|Au\|.$$

If $b_0 < 1$ then $A + B$ is also selfadjoint on $D(A)$.

For a proof see [7, Theorem V.4.3].

Lemma 5.2. *Let H_D and V be as above. Then $H_D + V$ is selfadjoint on $H^1(\mathbb{R}^3)^4$.*

Proof. Let $u \in H^1(\mathbb{R}^3)^4$. H_D is selfadjoint and V is symmetric. It follows from (5.3) and the Hardy inequality that

$$\|Vu\| \leq a\||x|^{-1}u\| + b\|u\| \leq 2a\|\nabla u\| + b\|u\|.$$

On the other hand, we see from (5.2) that $\|H_D u\|^2 = \|\nabla u\|^2 + m^2\|u\|^2$. Therefore, V is H_D -bounded, with H_D -bound $2a < 1$. Now the assertion follows from Theorem 5.1. \square

The selfadjointness of $(H_D + V)^2$ is clear. Let us consider the selfadjointness of S . Clearly, S is symmetric. Thus we have only to consider the m -accretivity of S .

Lemma 5.3 ([10]). *Let A and B be linear m -accretive operators in a Hilbert space H . Let D be a linear manifold invariant under $(1 + n^{-1}A)^{-1}$ for $n \in \mathbb{N}$. Assume that D is a core of B and there exist two constants $a, b \geq 0$ such that for all $u \in D_0 := (1 + A)^{-1}D$,*

$$0 \leq \operatorname{Re}(Au, Bu) + a \|u\|^2 + b \|Au\|^2.$$

If $b < 1$ then $A + B$ is also m -accretive in H .

Lemma 5.4. *Let H_D and V be as above. Then S is selfadjoint on $D(S)$.*

Proof. Let $u \in \mathcal{S}(\mathbb{R}^3)^4$, where $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space. Then we have

$$\begin{aligned} \operatorname{Re}((H_D + V)^2 u, \langle x \rangle^2 u) &= \operatorname{Re}((H_D + V)u, (H_D + V)(\langle x \rangle^2 u)) \\ &= \|\langle x \rangle(H_D + V)u\|^2 - 2 \operatorname{Im}((H_D + V)u, \alpha \cdot xu) \\ &\geq \|\langle x \rangle(H_D + V)u\|^2 - 2 \|\langle x \rangle(H_D + V)u\| \cdot \|u\| \\ &\geq -\|u\|^2. \end{aligned}$$

The assertion follows from Theorem 5.3. □

6. Applications to the Dirac equation

Let H_D and V be as in Section 5. In this section we consider, as an application of Theorem 1.1, the Cauchy problem for the Dirac equation:

$$(DE) \quad \begin{cases} i \frac{d}{dt} u = H(t)u + f(t) & \text{for } t \in (0, T), \\ u(0) = u_0 \end{cases}$$

in the Hilbert Space $X = L^2(\mathbb{R}^3)^4$, where $u_0 \in Y := H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$.

First we define $H(t)$ precisely. Let

$$\mathcal{H}(t) := H_D + V + q(t)I$$

with domain $D(\mathcal{H}(t)) = C_0^\infty(\mathbb{R}^3)^4$. $q(t)I$ is a maximal multiplication operator by $q(x, t)$, where $q(x, t) : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}$ is the time-dependent measurable real-valued potential. Furthermore, we impose $q(t)$ satisfying following conditions:

$$\begin{aligned} (q1) \quad & q(\cdot) \in L^1(0, T; \langle x \rangle L^\infty(\mathbb{R}^3)), \\ (q2) \quad & |\nabla q(\cdot)| \in L^1(0, T; L^\infty(\mathbb{R}^3)), \end{aligned}$$

where $\langle x \rangle L^\infty(\mathbb{R}^3) := \{\varphi \in L_{\text{loc}}^1(\mathbb{R}^3); \langle x \rangle^{-1} \varphi \in L^\infty(\mathbb{R}^3)\}$.

Since $\mathcal{H}(t)$ is symmetric, $\mathcal{H}(t)$ is closable. Then we take as $H(t)$ the closure $\tilde{\mathcal{H}}(t)$ of $\mathcal{H}(t)$, i.e., $H(t) = \tilde{\mathcal{H}}(t)$.

Let S be as in (5.1). Then S is selfadjoint on $D(S)$, with $S \geq 1$. Thus $Y = D(S^{1/2})$ is regarded as a Hilbert space, embedded continuously and densely in $L^2(\mathbb{R}^3)^4$, with inner product

$$(u, v)_{D(S^{1/2})} = (S^{1/2}u, S^{1/2}v), \quad u, v \in D(S^{1/2}).$$

Lemma 6.1. *Let S be as above. Then $D(S^{1/2}) = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$ and there exist positive constants c_1, c_2 such that*

$$(6.1) \quad c_1 \|S^{1/2}u\|^2 \leq \|u\|^2 + \|\nabla u\|^2 + \| |x|u \|^2 \leq c_2 \|S^{1/2}u\|^2, \quad u \in D(S^{1/2}).$$

Proof. Let $u \in D(S)$. Then we have

$$\begin{aligned} \|S^{1/2}u\|^2 &= (Su, u) \\ &= ((H_D + V)^2 u + u + |x|^2 u, u) \\ &= \|u\|^2 + \|(H_D + V)u\|^2 + \| |x|u \|^2. \end{aligned}$$

On the other hand, there exist positive constants c'_1, c'_2 such that

$$(6.2) \quad c'_1 (\|u\| + \|\nabla u\|) \leq \|u\| + \|(H_D + V)u\| \leq c'_2 (\|u\| + \|\nabla u\|).$$

Since $D(S)$ is a core for $S^{1/2}$, (6.1) holds for $u \in D(S^{1/2}) = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4$. \square

Now we shall verify conditions (I)–(IV) of Theorem 1.1.

Lemma 6.2. *Let $A(t) = iH(t)$ and S be as above. Assume that (q1), (q2) are satisfied. Then for each $T > 0$*

- (I) $\operatorname{Re}(A(t)v, v) = 0$, $v \in D(A(t))$, a.a. $t \in (0, T)$.
- (II) $Y = H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4 \subset D(A(t))$, a.a. $t \in (0, T)$.
- (III) *There exists $\beta \in L^1(0, T)$, $\beta \geq 0$ such that*

$$|\operatorname{Re}(A(t)u, Su)| \leq \beta(t) \|S^{1/2}u\|^2, \quad u \in D(S), \text{ a.a. } t \in (0, T).$$

- (IV) $A(\cdot) \in L_*^1(0, T; B(H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4, L^2(\mathbb{R}^3)^4))$.

Proof. Noting that $\operatorname{Re}(A(t)u, u) = -\operatorname{Im}(H(t)u, u)$, the assertion follows from symmetry of $H(t)$. Therefore, it is sufficient to show that there exist $\beta, \gamma \in L^1(0, T)$ such that

$$(6.3) \quad \|H(t)u\| \leq \gamma(t) \|S^{1/2}u\|, \quad u \in H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4, \quad \text{a.a. } t \in (0, T).$$

$$(6.4) \quad |\operatorname{Im}(H(t)u, Su)| \leq \beta(t) \|S^{1/2}u\|^2, \quad u \in D(S), \quad \text{a.a. } t \in (0, T).$$

First, we verify (6.3). It follows from condition (q1) that

$$\begin{aligned} \|H(t)u\| &\leq \|(H_D + V)u\| + \|q(t)u\| \\ &\leq \|(H_D + V)u\| + \gamma_q(t) \|\langle x \rangle u\|, \end{aligned}$$

where $\gamma_q \in L^1(0, T)$ depends on q . Thus we obtain (6.3).

Next, we verify (6.4). By integration by parts we have

$$\begin{aligned} \operatorname{Im}(H(t), Su) &= \operatorname{Im}((H_D + V)u, |x|^2 u) + \operatorname{Im}(q(t)u, (H_D + V)^2 u) \\ &= \operatorname{Re}((\alpha \cdot x)u, u) - \operatorname{Re}((\alpha \cdot \nabla q(t))u, (H_D + V)u). \end{aligned}$$

Hence it follows from the Cauchy-Schwarz inequality and condition (q2) that

$$\begin{aligned} |\operatorname{Im}(H(t), Su)| &\leq \| |x|u \| \cdot \|u\| + \| |\nabla q(t)|u \| \cdot \| (H_D + V)u \| \\ &\leq \| |x|u \| \cdot \|u\| + \beta_q(t) \|u\| \cdot \| (H_D + V)u \|, \end{aligned}$$

where $\beta_q \in L^1(0, T)$ depends on q . Therefore we obtain (6.4). \square

Assume further that

$$(f1) \quad f \in L^1(0, T; H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4).$$

Then we can apply Theorems 1.1 and 1.2 to conclude that the Dirac equation (DE) admits a unique solution $u \in W^{1,1}(0, T; L^2(\mathbb{R}^3)^4) \cap C(0, T; H^1(\mathbb{R}^3)^4 \cap H_1(\mathbb{R}^3)^4)$.

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